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Minimally non-diatonic pc-sets

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We discuss and enumerate pc-sets that are both not contained in any diatonic collection and are minimal with respect to this property, and we generalize this idea to other collections. We also consider related simplicial complexes and examine how some of their geometric properties reflect qualities of the associated pc-sets.

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1. Introduction

We describe pc-sets that are *minimally non-diatonic*, meaning that they are contained in no diatonic collection, but are minimal with respect to this property. For example, the pc-set 012 is minimally non-diatonic: no diatonic collection contains consecutive semitones, but removing any of the three pitch classes results in a diatonic set. By the Stanley-Reisner correspondence (see, for instance, Stanley 1996), these pc-sets correspond to the minimal non-faces of a simplicial complex. This complex has vertices corresponding to pitch classes, and faces corresponding to diatonic sets.

As simplicial complexes are especially convenient tools for encoding or visualizing data geometrically, they have appeared often in mathematical music theory. Tymoczko has explored polytopal complexes (a generalization of simplicial complexes) whose vertices correspond to chords, and whose faces encode voice leadings and relationships between these chords (see Tymoczko 2004, 2010). Bigo et al. (2013) construct simplicial complexes whose vertices are pitch classes and whose facets are chords in a particular progression. In the case when all the chords considered in this construction are diatonic, this complex would be a subcomplex of the first simplicial complex we consider in Section 3. Additionally, many researchers have modeled parsimony in voice leading by using graphs whose vertices correspond to chords. For example, Douthett and Steinbach (1998) consider graphs whose vertices correspond to triads. These graphs can be reframed as facet-adjacency graphs of complexes similar to the ones we discuss here.

Guerino Mazzola has also used simplicial complexes extensively in musical contexts, often with connections to algebraic geometry, see Mazzola (2002). In addition, Mazzola has observed that the set of triads in a given key, viewed as the faces of a simplicial complex on the key's pitch classes, triangulates a Möbius strip. We mention this example in a bit more depth in Section 3.

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Furthermore, simplicial complexes are at least implicitly present in some standard constructions: For example, the classical Tonnetz is the universal cover for a triangulated torus (see Munkres 2000 for definitions related to this observation, as well as Yust 2020 for further results). These Möbius strips and torii are also subcomplexes of the complex we consider.

2. Minimally non-diatonic sets

Throughout, we use integer notation, identifying the twelve pitch classes of the chromatic scale with elements of the set $\{0, 1, 2, \ldots, 9, \mathrm{T}, \mathrm{E}\}$ and setting 0 to be the pitch class corresponding to C. We call a pc-set *non-diatonic* if it is not contained in the set of pitch classes corresponding to any major key. For example, the collection of all twelve pitch classes is non-diatonic, as is the whole-tone scale 02468T. On the other hand, the pc-set 015 is *diatonic*, as it is contained in many diatonic collections (C sharp, for example).

Note that the non-diatonic property is, by definition monotonic with respect to supersets: If X is non-diatonic and X is a subset of Y, then Y is non-diatonic as well. This suggests the following definition.

Definition 2.1 A pc-set X is *minimally non-diatonic* if it is non-diatonic, but any proper subset of it is diatonic. Equivalently, X is minimally non-diatonic if it is non-diatonic, but the removal of any pitch class from X yields a diatonic pc-set.

From the definition, it is clear that every non-diatonic pc-set must contain at least one minimally non-diatonic pc-set.

The chromatic scale is non-diatonic, but it is far from being minimally non-diatonic, as one must remove at least five pitch classes before the resulting pc-set is diatonic. The pc-set 012 *is* minimally non-diatonic, though, as 01, 02, and 12 are all diatonic sets. Specifically, we have the following:

Observation 2.2 Up to prime form, there are six minimally non-diatonic pc-sets:

$$012, 014, 048, 0167, 0268, \text{ and } 0369.$$

Thus any non-diatonic pc-set must contain a transposition and/or inversion of at least one of the above sets. For example, the whole-tone pc-set 02468T contains the sets 048 and 0268.

It also follows that any pc-set of eight or more pitch classes must contain at least one of the minimally non-diatonic sets.

We originally performed the computation of Observation 2.2 using the computer algebra system Macaulay 2 (Grayson and Stillman n.d.), but the following approach (which is more mathematically pleasing, and more easily generalizable) was pointed out to us by Guerino Mazzola: For two pitch classes x and y, let $d(x, y)$ denote the clockwise distance between x and y on the circle of fifths. That is, $d(x, y)$ is the number of clockwise steps one needs to take on the circle of fifths to go from x to y. For example, $d(2, \mathrm{E}) = 3$ and $d(\mathrm{E}, 2) = 9$. In general, $d(y, x) = 12 - d(x, y)$.

To make notation easier, we only consider pc-sets here containing C (or 0). Now consider such a pc-set $X = \{0, x_1, x_2, \ldots, x_t\}$, listed in clockwise order around the circle of fifths, starting at 0. We define a vector $d(X)$ as follows.

$$d(X) = \langle d(0, x_1), d(x_1, x_2), \ldots, d(x_{t-1}, x_t), d(x_t, 0)\rangle.$$

For example, $d(\{0, 1, \mathrm{T}\}) = \langle 7, 3, 2\rangle$, which we explain in more detail here. Traverse the circle of fifths clockwise from 0 to reach 1: 0, 7, 2, 9, 4, E, 6, 1. This requires seven steps. Then go from

1 to T: $1, 8, 3, \mathrm{T}$, which takes 3 steps. Finally, go from T back to 0: $\mathrm{T}, 5, 0$, which requires two steps, and so we obtain the vector above. Note that for $d(X)$ to be defined, we must write X in clockwise order around the circle of fifths.

The following properties of the vector $d(X)$ are immediate:

(1) The sum of the entries of $d(X)$ is 12.
(2) Since every major scale appears as a consecutive sequence of seven pitch classes around the circle of fifths, X is diatonic if and only if it is contained in one of these sequences of seven consecutive pitch classes. This happens if and only if some entry of $d(X)$ is greater than or equal to 6. Thus, X is non-diatonic if and only if every entry of $d(X)$ is strictly less than 6.
(3) Finally, X is minimally non-diatonic if and only if it is non-diatonic (no entry of $d(X)$ exceeds 5), but the sum of any two consecutive entries (including the first and the last entries) is greater than or equal to 6.

For example, the set $X = \{0, 1, \mathrm{T}\}$ discussed above has $d(X) = \langle 7, 3, 2\rangle$, and since the first component is greater than or equal to 6, the set X is diatonic. Indeed, it is contained in the diatonic collection $\{6, 1, 8, 3, \mathrm{T}, 5, 0\}$, which occurs consecutively around the circle of fifths.

Given the above, we can classify all minimally non-diatonic pc-sets containing C by searching for the corresponding vectors $d(X)$. Such vectors must have all entries less than 6, must sum to 12, and the sum of any two consecutive entries must be 6 or more. For example, the vector $\langle 3, 5, 4\rangle$ satisfies these criteria, and the corresponding pc-set is 089: ascend three fifths from 0 to obtain 9, and ascend five fifths from 9 to obtain 8. This pc-set appears in our list, as its prime form is 014.

The same approach can be used to classify minimally non-pentatonic pc-sets as well, where *minimally non-pentatonic* pc-sets are defined in the obvious way. In particular, we have the following analogous properties to the above (with $d(X)$ defined as before):

(1) Since every pentatonic scale appears as a consecutive sequence of five pitch classes around the circle of fifths, X is pentatonic if and only if some entry of $d(X)$ is greater than or equal to 8. Thus, X is non-pentatonic if and only if every entry of $d(X)$ is strictly less than 8.
(2) X is minimally non-pentatonic if and only if no entry of $d(X)$ exceeds 7 and the sum of any two consecutive entries is greater than or equal to 8.

The classification of minimally non-pentatonic sets is then immediate: The only such vectors $d(X)$ would be $\langle 6, 6\rangle$, $\langle 5, 7\rangle$, $\langle 7, 5\rangle$, and $\langle 4, 4, 4\rangle$. The first corresponds to the tritone, the second and third to the semitone, and the fourth to the augmented triad. Thus, we have the following:

Observation 2.3 Up to prime form, there are three minimally non-pentatonic pc-sets:

$$01, 06, \text{and } 048.$$

We now discuss another way to understand the construction of minimal sets not contained in a given collection and its transpositions. Let X be a pc-set, and let

$$X, X + 1, X + 2, \ldots, X + t$$

denote all its distinct transpositions (where $X + i$ denotes the operation of adding i to every pitch class in X). If we write Y^c for the complement of a pc-set Y, then the minimally non-X pc-sets are exactly the sets A such that

$$A \cap (X + i)^c \neq \emptyset$$

for all i, and for each $a \in A$ there is an i with $A \cap (X + i)^c = \{a\}$. In most cases, finding the minimally non-X sets in this way is not very convenient. But in a few special cases, it can

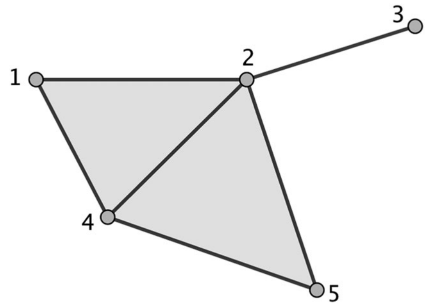


Figure 1. A geometric realization of the complex X.

be illuminating. In particular, if the sets $(X+i)^c$ are all distinct, then finding minimally non-X pc-sets is straightforward.

As an example of this method, consider the whole tone scale $X = 02468\text{T}$ and its one distinct transposition $X+1 = 13579\text{E}$. Then $X^c = X+1$ and $(X+1)^c = X$. So to find a minimally non-whole-tone pc-set, we would just select one pitch class from X and one from $X+1$, meaning the minimally non-whole-tone pc-sets are exactly the pairs of pitch classes whose parities differ.

For a slightly more involved example, consider Messiaen's third mode of limited transposition $X = 0234678\text{TE}$ (see Messiaen 1956). There are four distinct transpositions of X^c:

$$159, 26\text{T}, 37\text{E}, 048.$$

As these pc-sets are pairwise disjoint, any pc-set minimally not contained in this mode consists of a choice of exactly one pitch class from each of these sets (for example, 1678). In particular, all such pc-sets are tetrachords.

3. The simplicial complex of diatonic sets

For more detail on the basic definitions in this section (e.g. *simplicial complex, homology,* etc.), we refer the reader to Munkres (2000), although most online resources or standard books on combinatorial topology will cover these topics.

For our purposes, an *abstract simplicial complex* X with the vertex set V is a collection of subsets of V (known as *faces*), closed under inclusion (that is, if $A \in X$ and $B \subseteq A$, then $B \in X$). The faces that are maximal with respect to inclusion are known as the *facets*. For example, the set $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{2,5\}, \{4,5\}, \{1,2,4\}, \{2,4,5\}\}$ is a simplicial complex with three facets: 124, 23, and 245 (here and in what follows, we abuse notation by writing sets as words).

It is a basic fact in topology that every simplicial complex has a *geometric realization*, which is a geometric simplicial complex whose faces are given by those in the corresponding abstract simplicial complex. See Figure 1 for a geometric representation of the abstract simplicial complex X. Thus, in describing the complex Δ below, we are implicitly describing a geometric object as well. Note that the dimension of a face in the geometric realization is one less than the number of vertices in that face (e.g. the face 23 has dimension one).

One can describe an abstract simplicial complex by listing its faces, but one can also describe it by listing its facets, or by listing its minimal non-faces. For example, knowing the facets of the complex above uniquely determines the complex. Similarly, we can also list the minimal non-faces of the complex above as 13, 15, 34, and 35. These minimal non-faces determine the complex, as any subset of $\{1,2,3,4,5\}$ not containing any of these minimal non-faces must be a face of the complex. If f_i is the number of i-dimensional faces of a complex (so that $f_{-1}=1$, counting the empty set), the vector $(f_{-1}, f_0, f_1, \ldots)$ is known as the *f-vector* of the complex. The above complex has f-vector $(1,5,6,2)$.

Implicit in the our earlier construction of minimally non-diatonic pc-sets is a simplicial complex whose faces correspond to diatonic pc-sets, and whose minimal non-faces correspond to the minimally non-diatonic pc-sets. Formally, we define the complex Δ as follows.

Definition 3.1 The complex Δ has vertex set $\{0,1,2,\ldots,9,\mathrm{T},\mathrm{E}\}$ and faces corresponding to diatonic pc-sets. Thus, Δ is pure of dimension six and has twelve facets, one for each major key.

The question of the topology of Δ is an interesting one; while many "naturally occurring" complexes in combinatorics and commutative algebra are homotopy equivalent to bouquets of spheres, Δ is not. In fact, Δ only has non-vanishing reduced homology in dimension two. Computations show that the rank of the second homology group of Δ is 5.

We used Macaulay 2 to compute the f-vector of Δ as

$$1, 12, 66, 180, 240, 180, 72, 12,$$

and thus the *reduced Euler characteristic* of Δ (i.e. the topological invariant given by the alternating sum of the f-vector) is

$$\tilde{\chi}(\Delta) = 5.$$

Note that the complex Δ has minimal non-faces given by all transpositions and inversions of the minimally non-diatonic sets discussed in Observation 2.2. There are 12 minimal non-faces corresponding to 012, 24 corresponding to 014, 4 corresponding to 048, 6 corresponding to 0167, 6 corresponding to 0268, and 3 corresponding to 0369.

The complex's f-vector alone has some interesting information; for example it shows that there are exactly 180 diatonic trichords. This number can be computed by brute force as well: There are $\binom{12}{3} = 220$ total trichords, and there are $12 + 24 + 4 = 40$ non-diatonic trichords (see preceding paragraph).

Now we discuss some examples of how other properties of diatonic pc-sets can be interpreted in terms of the complex Δ:

Call two facets F, F' of a complex *adjacent* if there are vertices x and y such that $(F - x) \cup y = F'$. In other words, adjacent facets intersect in codimension-1 faces.

Recall that a complex is *strongly facet-connected* if for any two of its facets F and F', there exists a sequence $F = F_1, F_2, \ldots, F_t = F'$ such that F_i and F_{i+1} are adjacent for all i. It is immediate from this definition that a strongly facet-connected complex must be pure (meaning all its facets are of the same dimension).

Observation 3.2 The complex Δ is strongly facet-connected; a sequence of adjacent facets is given by traversing the circle of fifths:

024579E, 024679E, 124679E, 124689E, 134689E, 13468TE

13568TE, 013568T, 013578T, 023578T, 023579T, 024579T

The fact that Δ is strongly facet connected is something musicians already know: that it is possible to go from any diatonic key to any other by changing one pitch class at a time.

Recall that the *i-skeleton* of a complex Σ, for which we write $\Sigma_{\leq i}$, is the complex consisting of all faces of Σ of dimension less than or equal to i. For the complex X from Figure 1, the 1-skeleton $X_{\leq 1}$ consists of the four vertices and the four edges 12, 13, 14, and 23.

The following is a standard observation in combinatorial topology:

PROPOSITION 3.3 *Let Σ be a strongly facet-connected complex. Then any skeleton $\Sigma_{\leq i}$ is strongly facet-connected as well.*

The proof is straightforward, but we sketch it here: Suppose f and f' are two facets of $\Sigma_{\leq i}$ (meaning they are i-dimensional), and let them be contained in facets F and F', respectively. Since Σ is strongly facet-connected, there is a sequence of facets $F = F_0, F_1, F_2, \ldots, F_t = F'$ such that F_j and F_{j+1} intersect in a codimension-1 facet for each j. As the i-skeleton of F_j, $(F_j)_{\leq i}$, is strongly facet-connected for each j, the result follows.

Using Observation 3.2 and Proposition 3.3, we see that $\Delta_{\leq i}$ is strongly facet-connected. The connection to voice leadings is immediate:

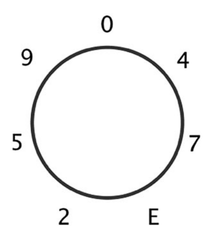
Observation 3.4 For any two diatonic pc-sets A and B, each of size k, there is a sequence of diatonic sets

$$A = A_0, A_1, \ldots, A_t = B$$

with $|A_i \cap A_{i+1}| = k - 1$ for all i.

While the circle of fifths order in Observation 3.2 shows that Δ (and any skeleton of Δ) is strongly facet-connected, the same reasoning shows that the complex of pentatonic collections (see Observation 2.3) is also strongly facet-connected, as are all of its skeleta.

In Mazzola (2016), Mazzola considers the simplicial complex of triads contained within a particular diatonic collection and observes that this complex triangulates a Möbius strip. For example, in the case of the key of C, this complex has vertex set $\{0, 2, 4, 5, 7, 9, \mathrm{E}\}$ and facets 047, 259, 47E, 059, 27E, 049, and 25E. This complex can be treated with the same techniques we used in obtaining Observations 2.2 and 2.3. Call this complex M, and consider the seven pitch classes written around a circle as follows:



Note that the facets of the complex M are exactly the sequences of three consecutive pitch classes when written around the circle above. The same reasoning used earlier recovers the fact that the complex M is strongly facet-connected, and also allows us to find the minimally non-M pc-sets: They are exactly the pairs of pitch classes separated by a tone or a semitone.

Other musical properties can be phrased in terms of Δ, our complex of diatonic pc-sets. Recall that if K is a face of a simplicial complex Σ, the *link* of K is the complex with faces

$$\{A \in \Sigma : A \cap K = \emptyset \text{ and } A \cup K \in \Sigma\}.$$

It is easily seen that the link of a pc-set $K \in \Delta$ consists of those pc-sets that co-exist in a key with K. For example, the link of 024 (the trichord containing C, D, and E) has facets 579E, 679E, and 579T, corresponding to the fact that this pc-set is contained in the keys of C, G, and F.

In the same way we built the simplicial complex Δ of diatonic sets, we could also build the simplicial complex of, say, pc-sets contained in some melodic minor collection. Computation with Macaulay 2 gives the following prime forms of all minimal non-melodic minor sets:

$$012, 0145, 0156, 0147, 0347, 0158, 0369, 02468\mathrm{T}.$$

It is worth noting the presence of a minimally non-melodic minor hexachord above, since all minimally non-diatonic sets are either trichords or tetrachords.

Inverting a major scale yields another major scale, and melodic minor scales behave similarly. As harmonic scales do not have this property, it makes more sense to alter our construction slightly for harmonic minor scales, considering a pc-set to be harmonic whenever it or its inversion is contained in a harmonic minor scale. Again computed with Macaulay 2, the minimal non-faces of this complex have prime forms

$$012, 0246, 0256, 0137, 0247, 0167, 0268, \text{ and } 03478.$$

This is, of course, just a short sampling of the many questions that can be asked about these complexes, and we believe other techniques developed for the study of simplicial complexes can likely be used to find combinatorially interesting or unique pc-sets.

Other collections prove not as nuanced. For example, the complex of whole-tone pc-sets is easily described. It has just two facets: 02468T and 13579E. As we noted earlier, the minimally non-whole tone pc-sets are easily described: They are the pairs of pitch classes with different parities.

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